XVII. *Of the Rectification of the Conic Sections. By the Rev.* John Hellins, *B. D. F. R. S. and Vicar of Potter's-Pury, in Northamptonshire.*

Read July 8, 1802.

PART I.

Of the Rectification of the Hyperbola: containing several new Series for that Purpose; together with the Methods of computing the constant Quantities by which the ascending Series differ from the descending ones.

INTRODUCTION.

The conic sections are a part of geometry so requisite in mensuration, in optics, astronomy, and other branches of natural philosophy, that the properties of these curves have been much studied in the course of the last hundred and fifty years; and there is hardly a writer on fluxions, of any note, who has not treated of their rectification. It may therefore seem, that little is now left to the industry of the present and future generations, in this part of the mathematics, but the proper application of theorems already investigated. Yet, while we admire the skill, and praise the industry, of those who have discovered new truths, or thrown new light on old ones, within that period, we shall do well to recollect, that it is now no more than one hundred and thirty-seven years, since the two great discoveries of *fluxions* and *infinite series* were made by Sir Isaac Newton; and that the observation of the late Mr. Emerson, respecting

the state of fluxions in his time, is in a great measure applicable to it in ours; viz. " *If arts and sciences of many hundred* " *years standing receive daily improvements and additions, it can-* " *not be supposed that this most sublime art of all, found out but* " *yesterday, can be arrived at perfection all on a sudden. If this* " *art be so exceedingly useful and valuable, it certainly deserves* " *the pains and attention of the learned mathematicians.*"*—And indeed, whoever considers the great number of mathematical and physico-mathematical problems which are solved by means of fluxions and series only, the several different ways in which series may be applied to the solution of the same problem, the fewness of those who employ themselves at all about these abstract sciences, and the still smaller number of those who have skill, leisure, and resolution enough to attempt any improvement in them; I say, whoever duly considers these things, (even without making allowance for the want of patronage which the liberal arts have of late years experienced,) will see reason to think, that many ages must yet elapse, before this most sublime and extensively useful method of computation will receive all the improvements of which it is capable. He will perceive, that, of the large field opened by Sir ISAAC NEWTON, a considerable part is still covered with briars and thorns. He will have no doubt, that the mine is not yet exhausted, but that, although the first workers of it have carried away the largest and most brilliant diamonds, enough still remain to reward the labour of those who shall have the resolution to dig deeper, and the patience of those who shall yet carefully sift the rubbish which has been thrown up by former adventurers.

The subject of the following sheets was first offered to my

* Preface to his Fluxions.

consideration in the year 1770, when a problem requiring the rectification of a large portion of an equilateral hyperbola was proposed in a periodical work; which problem I then solved by means of descending series; but, for want of an easy method of correcting the fluent so found, I laid it aside for the exercise of maturer judgment. Afterwards, the subject was resumed at different times, as leisure permitted, and put into nearly the same form in which it now appears, in the year 1795; since which time, till now, the duties of my station, and unexpected occurrences, have left me no opportunity to revise my papers.

As the investigation of the following theorems is very obvious and easy, I thought it probable that they might have been discovered by some other person before me; yet, upon perusal of Dr. HUTTON's *mathematical and philosophical Dictionary*, lately published, I find but one of them. And since, in the compilation of that work, as the learned and industrious author professes, "*Not only most of the encyclopedias already extant, and the* "*various transactions of the learned societies throughout Europe,* "*have been carefully consulted, but also all the original works, of* "*any reputation, which have hitherto appeared upon these sub-* "*jects,*"* I therefore conclude that all the rest are new.

The subject of this Paper is naturally divided into three sections: the first containing the investigations of the several series; the second, the methods of computing the constant quantities by which the ascending series differ from the descending ones; and the third, examples of their use, by way of illustration. But, for more convenient reference, I have further divided it into articles, or minor sections.

* See Dr. HUTTON's Address to the public, on the publication of the first part of the *Dictionary* above mentioned, in 1795; and preface to the *Dictionary*.

SECT. I. *The Investigations of the several Series.*

1. That the following processes may not be incumbered with symbols, and that the rate of convergency of the series obtained therefrom may be the more obvious, let the transverse axis of any hyperbola be called $2a$, and the conjugate axis 2; (by which notation, any ratio that these two lines can possibly have to each other may be expressed;)* let the abscissa be called x, the corresponding ordinate to the axis, y, and the length of the curve from the vertex to the ordinate, z. Then, by the well-known property of the curve, we have $2ax + xx = aayy$; from which x is found $= a\sqrt{(1+yy)} - a$, and $\dot{x} = \frac{ay\dot{y}}{\sqrt{(1+yy)}}$, and $\dot{z} = \sqrt{(\dot{y}\dot{y} + \dot{x}\dot{x})} = \dot{y}\sqrt{\left(1 + \frac{aayy}{1+yy}\right)} = \frac{\dot{y}\sqrt{(1+yy+aayy)}}{\sqrt{(1+yy)}}$; which equation, by writing ee for $1 + aa$, will become $\dot{z} = \frac{\dot{y}\sqrt{(1+eeyy)}}{\sqrt{(1+yy)}}$.

2. Now, the fluent of the expression on the right-hand side of the last equation may be taken in different series, according as the numerator or denominator of it is converted into series, and according as 1, $eeyy$, or yy, is made the leading term. By converting the numerator, $\sqrt{(1+eeyy)}$, into series, making 1 the leading term, we get $\dot{z} = \frac{\dot{y}}{\sqrt{(1+yy)}} \times : 1 + \frac{eeyy}{2} - \frac{e^4y^4}{2.4} + \frac{3e^6y^6}{2.4.6} - \frac{3.5e^8y^8}{2.4.6.8}$, &c. and then, by taking the fluents of $\frac{\dot{y}}{\sqrt{(1+yy)}}$, $\frac{\dot{y}yy}{\sqrt{(1+yy)}}$, $\frac{\dot{y}y^4}{\sqrt{(1+yy)}}$, &c. and denoting them by A, B, C, &c. respectively, we shall have

* With respect to homogeneity, about which some have shown more scrupulosity than discernment, I shall add a few words in a subsequent part.

$$A = \text{H. L. of } \overline{y + \surd(1 + yy)},$$
$$B = \frac{y\surd(1+yy) - A}{2},$$
$$C = \frac{y^3\surd(1+yy) - 3B}{4},$$
$$D = \frac{y^5\surd(1+yy) - 5C}{6},$$
$$E = \frac{y^7\surd(1+yy) - 7D}{8},$$
&c. &c.

And, lastly, by multiplying these quantities by their proper coefficients, we obtain (THEOREM I,)

$z = A + \frac{ee}{2}B - \frac{e^4}{2.4}C + \frac{3e^6}{2.4.6}D - \frac{3.5e^8}{2.4.6.8}E$, &c. where it is manifest that, unless the quantities B, C, D, &c. decrease in the ratio of ee to 1, the series will at last cease to converge; or, in other words, if yy be greater than $\frac{1}{ee}$, the terms of the series, at a great distance from the first, will diverge. And, of the nine theorems now produced, this is the only one that I have found in any other book.

3. But, by converting $\surd(1 + yy)$, the denominator of the fraction in the fluxionary equation in Art. I. into series, making 1 the leading term, we have $\dot{z} = \dot{y}\surd(1 + eeyy) \times : 1 - \frac{yy}{2} + \frac{3y^4}{2.4} - \frac{3.5y^6}{2.4.6} + \frac{3.5.7y^8}{2.4.6.8}$, &c. and, by taking the fluents of $\dot{y}\surd(1 + eeyy)$, $\dot{y}yy\surd(1 + eeyy)$, $\dot{y}y^4\surd(1 + eeyy)$, &c. and calling them A, B, C, &c. we shall have

$$A = \frac{y}{2}\surd(1 + eeyy) + \frac{1}{2e} \times \text{H, L. } \overline{ey + \surd(1 + eeyy)},$$
$$B = \frac{y(1 + eeyy)^{\frac{3}{2}} - A}{4ee},$$
$$C = \frac{y^3(1 + eeyy)^{\frac{3}{2}} - 3B}{6ee},$$
$$D = \frac{y^5(1 + eeyy)^{\frac{3}{2}} - 5C}{8ee},$$
$$E = \frac{y^7(1 + eeyy)^{\frac{3}{2}} - 7D}{10ee},$$
&c. &c.

And then, by multiplying these quantities by their respective coefficients, we obtain (THEOREM II,)

$$z = A - \frac{1}{2}B + \frac{3}{2.4}C - \frac{3.5}{2.4.6}D + \frac{3.5.7}{2.4.6.8}E, \text{ \&c.}$$

which series will converge till y becomes greater than 1; and consequently is a better series than that above found, which ceases to converge when y becomes greater than $\frac{1}{e}$. But, when y is much greater than 1, each of these series will diverge very swiftly; and, notwithstanding they are of that form which admits of a transformation to others which will converge, still, even by that means, their values will not be obtained without great labour. But here we shall have the pleasure of finding series which will quickly answer the purpose. For,

4. By converting the denominator, $\surd(yy + 1)$, into series, making yy the leading term, we get $\dot{z} = \dot{y}\surd(eeyy + 1) \times : \frac{1}{y} - \frac{1}{2y^3} + \frac{3}{2.4y^5} - \frac{3.5}{2.4.6y^7} + \frac{3.5.7}{2.4.6.8y^9}$, &c. And here, again, by denoting the fluents of $\frac{\dot{y}\surd(eeyy+1)}{y}$, $\frac{\dot{y}\surd(eeyy+1)}{y^3}$, $\frac{\dot{y}\surd(eeyy+1)}{y^5}$, &c. by A, B, C, &c. respectively, we shall have $A = \surd(eeyy + 1) + \text{H. L. } \frac{\surd(eeyy+1) - 1}{ey}$,

$$B = \frac{-(eeyy+1)^{\frac{3}{2}}}{2yy} + \frac{eeA}{2},$$

$$C = \frac{-(eeyy+1)^{\frac{3}{2}}}{4y^4} - \frac{eeB}{4},$$

$$D = \frac{-(eeyy+-)^{\frac{3}{2}}}{6y^6} - \frac{3eeC}{6},$$

$$E = \frac{-(eeyy+1)^{\frac{3}{2}}}{8y^8} - \frac{5eeD}{8},$$

&c. &c.

and then, (THEOREM III,)

$$z = \begin{cases} A - \frac{1}{2}B + \frac{3}{2.4}C - \frac{3.5}{2.4.6}D + \frac{3.5.7}{2.4.6.8}E, \text{ \&c.} \\ -d: \end{cases}$$

which series will converge the swifter the greater y is in comparison of 1, but will diverge when y is less than 1. It also wants a correction, (here denoted by the letter d,) which shall be given in its due place. This series then, when y becomes great in comparison of 1, will converge very swiftly, and becomes useful in those cases where the ascending series above investigated fail.

But, since the value of z may be expressed in another descending series, it will be proper to consider that also.

5. The expression $\frac{\dot{y}\sqrt{(1+eeyy)}}{\sqrt{(1+yy)}}$ is evidently $= \frac{e\dot{y}y}{\sqrt{(1+yy)}}\sqrt{(1+\frac{1}{eeyy})}$, which, by converting $\sqrt{(1+\frac{1}{eeyy})}$ into series, making 1 the leading term, becomes $\frac{e\dot{y}y}{\sqrt{(1+yy)}} \times : 1 + \frac{1}{2eeyy} - \frac{1}{2.4e^4y^4} + \frac{3}{2.4.6e^6y^6} - \frac{3.5}{2.4.6.8e^8y^8}$, &c. Here, the fluent of $\frac{e\dot{y}y}{\sqrt{(1+yy)}}$, the first term of the series, is $e\sqrt{(1+yy)}$; and, calling the fluents of $\frac{\dot{y}}{y\sqrt{(1+yy)}}$, $\frac{\dot{y}}{y^3\sqrt{(1+yy)}}$, $\frac{\dot{y}}{y^5\sqrt{(1+yy)}}$, &c. A, B, C, &c. respectively, we have

$$A = \text{H. L.}\ \frac{\sqrt{(yy+1)}-1}{y},$$
$$B = \frac{-\sqrt{(yy+1)}}{2yy} - \frac{A}{2},$$
$$C = \frac{-\sqrt{(yy+1)}}{4y^4} - \frac{3B}{4},$$
$$D = \frac{-\sqrt{(yy+1)}}{6y^6} - \frac{5C}{6},$$
&c. &c.

and thence, (THEOREM IV,)

$$z = \begin{cases} e\sqrt{(yy+1)} \\ +\frac{1}{2e}A - \frac{1}{2.4e^3}B + \frac{3}{2.4.6e^5}C - \frac{3.5}{2.4.6.8e^7}D, \text{ \&c.} \\ -d: \end{cases}$$

which series will converge the swifter, the greater y is in com-

parison of 1, and has an evident advantage over the last, in that it converges by the powers of ee, as well as by those of yy; so that its convergency will not cease, till the quantities B, C, D, &c. increase in the ratio of 1 to ee, that is, when y becomes equal to, or less than, $\frac{1}{e}$. This series, therefore, will be very useful for the greatest part of the hyperbola, when it is corrected by the constant quantity here denoted by d, the value of which is attainable several ways, as will appear in the next section.

6. These four theorems, or indeed two of them only, are sufficient for the rectification of any portion whatever of any conical hyperbola. Yet, as I have discovered several other series for that purpose, which are more convenient in particular cases, and of which some are useful in computing the constant quantity above denoted by d, (by which the ascending series differ from the descending ones,) it may be proper now to give the investigations of them also.

7. Put $1 + eeyy = uu$; then will yy be $= \frac{uu-1}{ee}$, and $1 + yy = \frac{uu+ee-1}{ee} =$ (by the notation in Art. 1, where ee was put $= aa + 1$,) $\frac{uu+aa}{ee}$, and therefore $\surd(1+yy) = \frac{\surd(uu+aa)}{e}$, and thence $\surd\left(\frac{1+eeyy}{1+yy}\right) = \frac{eu}{\surd(uu+aa)}$. Moreover, $\dot{y}$ will be $= \frac{\dot{u}u}{e\surd(uu-1)}$, and we shall have $\dot{y}\surd\left(\frac{1+eeyy}{1+yy}\right) = \dot{z} = \frac{\dot{u}uu}{\surd(uu-1)\times\surd(uu+aa)} = \frac{\dot{u}u}{\surd(uu-1)\times\surd\left(1+\frac{aa}{uu}\right)} = \frac{\dot{u}u}{\surd(uu-1)} \times : 1 - \frac{aa}{2uu} + \frac{3a^4}{2.4u^4} - \frac{3.5a^6}{2.4.6u^6} + \frac{3.5.7a^8}{2.4.6.8u^8}$, &c. Now the fluent of $\frac{\dot{u}u}{\surd(uu-1)}$ is $\surd(uu-1)$; and, if the fluents of $\frac{\dot{u}}{u\surd(uu-1)}$, $\frac{\dot{u}}{u^3\surd(uu-1)}$, $\frac{\dot{u}}{u^5\surd(uu-1)}$, &c. are denoted by A, B, C, &c. respectively, we shall have

$A =$ circ. arch, rad. being 1, and sec. u,

$B = \frac{\sqrt{(uu-1)}}{2uu} + \frac{A}{2}$,

$C = \frac{\sqrt{(uu-1)}}{4u^4} + \frac{3B}{4}$,

$D = \frac{\sqrt{(uu-1)}}{6u^6} + \frac{5C}{6}$,

&c. &c.

And, lastly, by multiplying these quantities by their proper coefficients, and collecting the several terms in due order, we shall have (THEOREM V,)

$$z = \sqrt{(uu-1)} - \frac{aa}{2}A + \frac{3a^4}{2.4}B - \frac{3.5a^6}{2.4.6}C + \frac{3.5.7a^8}{2.4.6.8}D, \&c.$$

Here it is remarkable, that the terms $\frac{\sqrt{(uu-1)}}{2uu}$, $\frac{\sqrt{(uu-1)}}{4u^4}$, $\frac{\sqrt{(uu-1)}}{6u^6}$, &c. which enter into the values of B, C, D, &c. always decrease while y increases from o *ad infinitum*; and indeed decrease more swiftly than the terms of either of the descending series in the preceding articles; and therefore this series may be used for computing the length of any portion of the hyperbola. For although the terms of it, taken at a great distance from the first, will diverge by the powers of aa, when a is greater than 1, yet, as the signs of these terms are alternately + and —, it admits of an easy transformation into another series, which will always converge by the powers of $\frac{aa}{1+aa}$. It also wants no correction; in consequence of which it has a peculiar use, which will appear in the next section.

8. But the fluxionary expression $\frac{\dot{u}\,uu}{\sqrt{(uu-1)} \times \sqrt{(uu+aa)}}$, obtained in the preceding Art. is $= \frac{\dot{u}\,u}{\sqrt{(uu+aa)} \times \sqrt{(1-\frac{1}{uu})}} = \frac{\dot{u}\,u}{\sqrt{(uu+aa)}} \times :$ $1 + \frac{1}{2uu} + \frac{3}{2.4u^4} + \frac{3.5}{2.4.6u^6} + \frac{3.5.7}{2.4.6.8u^8}$, &c. Here the fluent

of $\frac{\dot{u}u}{\sqrt{(uu+aa)}}$ is $\sqrt{(uu+aa)}$; and, if the fluents of $\frac{\dot{u}}{u\sqrt{(uu+aa)}}$, $\frac{\dot{u}}{u^3\sqrt{(uu+aa)}}$, $\frac{\dot{u}}{u^5\sqrt{(uu+aa)}}$, &c. are denoted by A, B, C, &c. we shall have

$$A = \frac{1}{a} \text{ H. L. } \frac{\sqrt{(uu+aa)}-a}{u},$$
$$B = \frac{-\sqrt{(uu+aa)}}{2aa\,uu} - \frac{A}{2aa},$$
$$C = \frac{-\sqrt{(uu+aa)}}{4aa\,u^4} - \frac{3B}{4aa},$$
$$D = \frac{-\sqrt{(uu+aa)}}{6aa\,u^6} - \frac{5C}{6aa},$$
&c. &c.

And, by multiplying these quantities by their proper coefficients, and collecting the products together, we shall have (Theorem VI,)

$$z = \begin{cases} \sqrt{(uu+aa)} + \frac{1}{2}A + \frac{3}{2.4}B + \frac{3.5}{2.4.6}C + \frac{3.5.7}{2.4.6.8}D, \&c. \\ -d. \end{cases}$$

Here also, the terms $\frac{\sqrt{(uu+aa)}}{2aa\,uu}$, $\frac{\sqrt{(uu+aa)}}{4aa\,u^4}$, $\frac{\sqrt{(uu+aa)}}{6aa\,u^6}$, &c. which are component parts of this series, always decrease while y increases from 0 *ad infinitum;* and therefore the length of any portion whatever of the hyperbola may be computed by this series also, when the value of the constant quantity d, to be taken from it, is known. But the case to which this theorem ought to be applied is, when y is equal to, or greater than 1. And it has an advantage over some of the descending series, in that the terms $\frac{A}{2}$, $\frac{3}{4}B$, $\frac{5}{6}C$, &c. are divided by aa, as will appear in the use of it.

9. When $a = 1$, that is, when the hyperbola is equilateral, the fluxionary equation in Article 7 becomes $\dot{z} = \frac{\dot{u}\,uu}{\sqrt{(uu-1)} \times \sqrt{(uu+1)}}$

$$= \frac{\dot{u}\,uu}{\sqrt{(u^4-1)}} = \frac{\dot{u}}{\sqrt{\left(1-\frac{1}{u^4}\right)}} = \dot{u} + \frac{\dot{u}}{2u^4} + \frac{3\dot{u}}{2.4u^8} + \frac{3.5\dot{u}}{2.4.6^{12}} + \frac{3.5.7\dot{u}}{2.4.6.8u^{16}},$$

&c.; the correct fluents of which are (THEOREM VII,)

$$z = \begin{cases} u - \frac{1}{2.3u^3} - \frac{3}{2.4.7u^7} - \frac{3.5}{2.4.6.11u^{11}} - \frac{3.5.7}{2.4.6.8.15u^{15}}, \text{ \&c.} \\ -d. \end{cases}$$

Which series is better adapted to this case than either of the preceding ones, in that it is much simpler, and converges twice as fast. And the correction of it is easily attainable by various methods.

10. But the original fluxionary equation in Art. 1, admits of a conversion into series, two different ways from any of those which have yet been taken. For, by the Binomial theorem, $\dot{y}\sqrt{\left(1+\frac{aa\,yy}{1+yy}\right)}$ is $= \dot{y} + \frac{aa}{2} \cdot \frac{\dot{y}\,yy}{1+yy} - \frac{a^4}{2.4} \cdot \frac{\dot{y}\,y^4}{(1+yy)^2} + \frac{3a^6}{2.4.6} \cdot \frac{\dot{y}\,y^6}{(1+yy)^3} - \frac{3.5a^8}{2.4.6.8} \cdot \frac{\dot{y}\,y^8}{(1+yy)^4}$, &c. where, putting A, B, C, &c. for the fluents of $\frac{\dot{y}\,yy}{1+yy}$, $\frac{\dot{y}\,y^4}{(1+yy)^2}$, $\frac{\dot{y}\,y^6}{(1+yy)^3}$, &c. respectively, we have

$$A = y - \text{circ. arch, rad. } 1, \text{and tang. } y,$$

$$B = -\frac{y^3}{2\,(1+yy)} + \frac{3A}{2},$$

$$C = -\frac{y^5}{4\,(1+yy)^2} + \frac{5B}{4},$$

$$D = -\frac{y^7}{6\,(1+yy)^3} + \frac{7C}{6},$$

&c. &c.

and thence (THEOREM VIII,)

$$z = y + \frac{aa}{2}A - \frac{a^4}{2.4}B + \frac{3a^6}{2.4.6}C - \frac{3.5a^8}{2.4.6.8}D, \text{ \&c.}$$

In which series, it is pretty evident, the quantities A, B, C, D, &c. will have a convergency while y increases from 0 *ad infinitum*, although the convergency will be but slow after y becomes greater than 1. It is obvious too, that this series

vanishes together with y, and therefore needs no correction. And for this reason chiefly I have introduced it, as it affords us another mean of obtaining the value of the constant quantity d, by which the descending series are to be corrected.

11. But the fluxionary expression $\dot{y}\sqrt{\left(\frac{1+eeyy}{1+yy}\right)}$, obtained in Art. 1, is evidently $= \dot{y}\sqrt{\left(ee + \frac{1-ee}{1-yy}\right)} =$ also to $\dot{y}\sqrt{\left(ee - \frac{aa}{1+yy}\right)}$; and this expression converted into series, by the Binomial theorem, becomes $e\dot{y} - \frac{aa\dot{y}}{2e(1+yy)} - \frac{a^4\dot{y}}{2.4e^3(1+yy)^2} - \frac{3a^6\dot{y}}{2.4.6e^5(1+yy)^3} - \frac{3.5a^8\dot{y}}{2.4.6.8e^7(1+yy)^4}$, &c. Here again, denoting the fluents of $\frac{\dot{y}}{1+yy}$, $\frac{\dot{y}}{(1+yy)^2}$, $\frac{\dot{y}}{(1+yy)^3}$, &c. by A, B, C, &c. we shall have

$$A = \text{circ. arch, rad. } 1\text{, and tang. } y,$$
$$B = \frac{y}{2(1+yy)} + \frac{A}{2},$$
$$C = \frac{y}{4(1+yy)^2} + \frac{3B}{4},$$
$$D = \frac{y}{6(1+yy)^3} + \frac{5C}{6},$$
&c. &c.

And, by proceeding as before directed, we get (THEOREM IX,)

$$z = ey - \frac{aa}{2e}A - \frac{a^4}{2.4e^3}B - \frac{3a^6}{2.4.6e^5}C - \frac{3.5a^8}{2.4.6.8e^7}D, \text{ \&c.}$$

And this series, it is obvious, will converge the swifter the greater y is, so that it will begin to converge swiftly when the preceding series begins to converge slowly. It is evident too, that this series vanishes together with y, and therefore wants no correction. Moreover, it has an advantage over the preceding series, in that the coefficients of it decrease by the powers of $\frac{aa}{ee}$, that is, by $\frac{aa}{1+aa}$. And it supplies us with a different expression of the value of d, as will appear in the next section, to which I now proceed.

SECT. II. *The Methods of computing the Values of the constant Quantities by which the ascending Series differ from the descending ones.*

12. Now the methods of obtaining these constant quantities are such as are shewn in my *Mathematical Essays*, (published in 1788,) pages 100, 101, 102, &c. to 112; viz. either by computing the value of both an ascending and a descending series, taking for y some small definite quantity, or by comparing the values of those series together when y is taken immensely great: the former of which methods is more general, but the latter, when it can be applied, commonly affords the easiest computation. In this section, I shall make use of both these methods, as the one or the other is best suited to the case in hand. I begin with the use of the latter method, in comparing together all the different expressions of the value of z, which are reduced to few terms in the case when y becomes immensely great.

Now, when y is taken immensely great, the value of z in THEOREM III. Art. 4, becomes barely $= ey - d$. For, in this case, the H. L. $\frac{\sqrt{(eeyy+1)}-1}{ey}$ becomes the logarithm of the ratio of equality, which is $= 0$. And then A is barely $= \sqrt{(eeyy+1)} + 0 = ey + \frac{1}{2ey} - \frac{1}{2.4e^3y^3}$, &c. all which terms, after the first, vanish in this case; and therefore eeA, which occurs in the value of B, becomes barely e^3y. Moreover, the radical expression $\frac{-(eeyy+1)^{\frac{3}{2}}}{2yy}$, which enters into the value of B, becomes barely $= \frac{-e^3y}{2}$; and thence we have $\text{B} = \frac{-e^3y+e^3y}{2} = 0$. And, since each of the expressions $\frac{(eeyy+1)^{\frac{3}{2}}}{4y^4}$, $\frac{(eeyy+1)^{\frac{3}{2}}}{6y^6}$, &c. evidently becomes $= 0$, in this case, and since B has been

shown to be $=0$, it will thence follow, that all the terms denoted by C, D, E, &c. will vanish, and there will be left $z = ey - d$.

13. And in like manner it will appear, that the value of z in Theorem IV, Art. 5, when y becomes immensely great, is also $= ey - d$. For, in this case, H. L. $\frac{\sqrt{(yy+1)}-1}{y}$ becomes $= 0$; and each of the expressions $\frac{\sqrt{(yy+1)}}{2yy}$, $\frac{\sqrt{(yy+1)}}{4y^4}$, &c. also becomes $= 0$; and, consequently, $z = \begin{cases} e\sqrt{(yy+1)} \\ -d. \end{cases}$ But, since e is a finite quantity, the expression $e\sqrt{(yy+1)} = ey + \frac{e}{2y} - \frac{e}{2.4y^3}$, &c. when y is immensely great, becomes barely $= ey$. Therefore, in this case, we have $z = ey - d$.

14. Corollary. And hence it appears, that the series in these two theorems are equal to each other, and, consequently, that the constant quantity to be subtracted from each of them, by way of correction, is the same.

15. The first term of the series which expresses the value of z in Theorem V, Art. 7, is $\sqrt{(uu-1)}$, which, by the notation there used, is always $= ey$. And, when y becomes immensely great, the terms $\frac{\sqrt{(uu-1)}}{2uu}$, $\frac{\sqrt{(uu-1)}}{4u^4}$, $\frac{\sqrt{(uu-1)}}{6u^6}$, &c. which enter into the values of B, C, D, &c. vanish; but A becomes $=$ *the quadrantal arch of the circle, of which the radius is* 1; and thence we have $B = \frac{A}{2}$, $C = \frac{3}{4}B, = \frac{3}{2.4}A$, $D = \frac{5}{6}C = \frac{3\,5}{2.4.6}A$, &c. and these values being written for B, C, D, &c. in the series, we have, in this case, $z = ey - \frac{aa}{2}A + \frac{3a^4}{2.2.4}A - \frac{3.3.5a^6}{2.2.4.4.6}A + \frac{3.3.5.5.7a^8}{2.2.4.4.6.6.8}A$, &c. And, since this series always gives the correct value of z, we have now discovered the value of d, the constant quantity to be subtracted from the

descending series given in Theorems III and IV. The series to which d is $=$, viz. $A \times : \frac{aa}{2} - \frac{3a^4}{2.2.4} + \frac{3.3.5a^6}{2.2.4.4.6} - \frac{3.3.5.5.7a^8}{2.2.4.4.6.6.8}$, &c. will indeed diverge when a is greater than 1; yet, as was observed in Art. 7, it is of that form which admits of transformation into another which will always converge.

16. For the reasons above given in Articles 12 and 13, each of the terms A, B, C, &c. in Theorem VI, Art. 8, vanishes when y becomes immensely great, and z is then barely $= \left\{ \begin{array}{l} \surd(uu + aa) \\ - d. \end{array} \right.$ And, since $\surd(uu + aa)$ is, by the notation in Art. 1 and 7, $= \surd(ee\,yy + ee)$, which, in this case, becomes barely $= ey$, we have $z = ey - d$. Here we see that the series in this Theorem, and in Theorems III and IV, are always $=$ to each other, and consequently differ from each of the ascending series by the same constant quantity d, the value of which was discovered in the preceding Article.

17. When y becomes immensely great, the value of z in Theorem VII, Art. 9, becomes barely $= u - d$. And, since u is universally $= \surd(ee\,yy + 1)$, it will, in this case, be $= ey$; and we shall have $z = ey - d$, which is the very expression given by all the other descending series in the like case. But, when the hyperbola is equilateral, as was supposed in Art. 9, a is $= 1$, and we have $d = 1{\cdot}57079632 \times : \frac{1}{2} - \frac{3}{2.2.4} + \frac{3.3.5}{2.2.4.4.6} - \frac{3.3.5.5.7}{2.2.4.4.6.6.8}$, &c.

Moreover, when y is $= 0$, z is also $= 0$, and u is $= 1$; and therefore, by this theorem, we have $0 =$

$$\left\{ \begin{array}{l} 1 - \frac{1}{2.3} - \frac{3}{2.4.7} - \frac{3.5}{2.4.6.11} - \frac{3.5.7}{2.4.6.8.15}, \text{ \&c. or} \\ - d \end{array} \right.$$

$$d = 1 - \frac{1}{2.3} - \frac{3}{2.4.7} - \frac{3.5}{2.4.6.11} - \frac{3.5.7}{2.4.6.8.15}, \text{ \&c.}$$

And hence it follows, that this very slowly converging series is $= 1{\cdot}57079632 \times : \frac{1}{2} - \frac{3}{2.2.4} + \frac{3.3.5}{2.2.4.4.6} - \frac{3.3.5.5.7}{2.2.4.4.6.6.8}$, &c. by which expression its value is easily attainable, and will be found to be $= 0{\cdot}59907012$.

I observe, *in transitu*, that the ratio of this slowly converging series, $1 - \frac{1}{2.3} - \frac{3}{2.4.7} - \frac{3.5}{2.4.6.11} - \frac{3.5.7}{2.4.6.8.15}$, &c. to a series of good convergency, is easily attainable; by which mean we may likewise compute its value to any degree of exactness.

18. A general expression of the value of d being found in Art. 15, by which it may be computed, whatever be the ratio of the two axes of the hyperbola, I might now proceed to show the use of the theorems by a few examples; but, as the same series is attainable another way, and the same value of d is attainable also by different series, it will be no less curious than useful to show in what manner.

19. The nth term of the series of quantities $\frac{-y^3}{2(1+yy)}$, $\frac{-y^5}{4(1+yy)^2}$, $\frac{-y^7}{6(1+yy)^3}$, &c. which enter into the values of B, C, D, &c. in Theorem VIII, Art. 10, is evidently $\frac{-y^{2n+1}}{2n(1+yy)^n}$, which, by the Binomial theorem, is $= \frac{-y^{2n+1}}{2n} \times : y^{-2n} - ny^{-2n-2} + n \cdot \frac{n+1}{2} y^{-2n-4} - n \cdot \frac{n+1}{2} \cdot \frac{n+2}{2} y^{-2n-6}$, &c. $= -\frac{y}{2n} + \frac{1}{2y} - \frac{n+1}{4y^3}$, &c. which, when y becomes immensely great, is barely $= \frac{y}{2n}$. And the value of A, in this case, is y — *the quadrantal arch of a circle, of which the radius is* 1. Let this quadrantal arch be denoted by α; then, by substituting for A, B, C, &c. their proper values as they thus arise, we have

$$\begin{aligned}
A &= \ldots\ldots\ldots\ldots\ y - \alpha,\\
B &= -\frac{y}{2} + \frac{3y}{2} - \frac{3}{2}\alpha = y - \frac{3}{2}\alpha,\\
C &= -\frac{y}{4} + \frac{5y}{4} - \frac{3.5}{2.4}\alpha = y - \frac{3.5}{2.4}\alpha,\\
D &= -\frac{y}{6} + \frac{7y}{6} - \frac{3.5.7}{2.4.6}\alpha = y - \frac{3.5.7}{2.4.6}\alpha,\\
&\&c. \qquad \&c.
\end{aligned}$$

And, lastly, by writing these values of A, B, C, &c. in the Theorem, we have, in this case,

$$z = \begin{cases} y + \frac{aa}{2}y - \frac{a^4}{2.4}y + \frac{3a^6}{2.4.6}y - \frac{3.5a^8}{2.4.6.8}y, \ \&c. \\ -\frac{aa}{2}\alpha + \frac{3a^4}{2.2.4}\alpha - \frac{3.3.5a^6}{2.2.4.4.6}\alpha + \frac{3.3.5.5.7a^8}{2.2.4.4.6.6.8}\alpha, \ \&c. \end{cases}$$

But the series $y + \frac{aa}{2}y - \frac{a^4}{2.4}y + \frac{3a^6}{2.4.6}y - \frac{3.5a^8}{2.4.6.8}y$, &c. is evidently $= y\sqrt{(1 + aa)}$, which, by the notation in Art. 1, is $= ey$. We therefore have, in this case,

$$z = ey - \alpha \times : \frac{aa}{2} - \frac{3a^4}{2.2.4} + \frac{3.3.5a^6}{2.2.4.4.6} - \frac{3.3.5.5.7a^8}{2.2.4.4.6.6.8}, \ \&c.$$

And, since this theorem always gives the correct value of z, we have now the satisfaction of seeing a confirmation of the truth of our conclusion in Art. 15, by obtaining the very same expression by a very different process.

20. From what has been shewn in Articles 12, 13, &c. it will be very evident to any one who runs his eye over the component parts of the series given in Theorem IX, Art. 11, that, when y becomes immensely great, A becomes = *the quadrantal arch of a circle, of which the radius is* 1, which arch was denoted in the preceding Art. by α; and that

$$\begin{aligned}
B &= 0 + \frac{A}{2} = \frac{1}{2}\alpha,\\
C &= 0 + \frac{3B}{4} = \frac{3}{2.4}\alpha,\\
D &= 0 + \frac{5C}{6} = \frac{3.5}{2.4.6}\alpha,\\
&\&c. \qquad \&c.
\end{aligned}$$

And, these values being written for A, B, C, &c. in the Theorem, it gives, in this case,

$$z = ey - \frac{aa}{2e}\alpha - \frac{a^4}{2.2.4e^3}\alpha - \frac{3.3a^6}{2.2.4.46e^5}\alpha - \frac{3.3.5.5a^8}{2.2.4.4.6.6.8e^7}\alpha, \text{ \&c.}$$

And, since this Theorem also always gives the correct value of z, we shall, by comparing the expression now obtained with those which were found for z, in the like case, in Articles 12, 13, 15, 16, and 19, see that we have now got another general expression of the value of d, viz. $\alpha \times : \frac{aa}{2e} + \frac{a^4}{2.2.4e^3} + \frac{3.3a^6}{2.2.4.4.6e^5} + \frac{3.3.5.5a^8}{2.2.4.4.6.6.8e^7}$, &c. in which series ee is $= aa + 1$, and therefore it must always converge. Yet it should not be hastily concluded, that this expression of the value of d is always preferable to that which was obtained in Articles 15 and 19; for, when a is a large number, the powers of $\frac{aa}{ee} = \frac{aa}{1+aa}$, by which the series converges, will decrease very slowly.

21. However, when it happens that a is a large number, the value of d may be obtained by means of two series, which, in that case, will converge pretty swiftly; or indeed by means of three series, each of which will converge about twice as fast as either of the two series. But, for the sake of brevity, I shall at present describe the method of computing the value of d by two series only, and so conclude this section.

The series proper to be used on this occasion, it is obvious, are those which are given in Theorems II and IV, Articles 3 and 5; and the value of y to be assumed, is $\frac{1}{\sqrt{e}}$, with which value each of the series will have nearly the same rate of convergency. As this will best appear by an example, I will give one, taking $a = 7$. Now, with this value of a, we have $ee = aa + 1 = 50$, and $y = \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{50}} = 0{\cdot}141421356$; and, by

writing these values for e and y in Theorem II, Art. 3, we have

$$A = \frac{1}{2\sqrt{e}}\sqrt{(1+e)} + \frac{1}{2e}\text{H. L.}\left(\sqrt{e}+\sqrt{(1+e)}\right) = 0{\cdot}6547{,}320,$$

$$B = \frac{e^{-\frac{1}{2}}(1+e)^{\frac{3}{2}} - A}{4ee} = 0{\cdot}0398{,}409,$$

$$C = \frac{e^{-\frac{3}{2}}(1+e)^{\frac{3}{2}} - 3B}{6ee} = 0{\cdot}0036{,}665,$$

$$D = \frac{e^{-\frac{5}{2}}(1+e)^{\frac{3}{2}} - 5C}{8ee} = 0{\cdot}0003{,}853.$$

$$E = \frac{e^{-\frac{7}{2}}(1+e)^{\frac{3}{2}} - 7D}{10ee} = 0{\cdot}0000{,}434,$$

$$F = \frac{e^{-\frac{9}{2}}(1+e)^{\frac{3}{2}} - 9E}{12ee} = 0{\cdot}0000{,}051,$$

$$G = \frac{e^{-\frac{11}{2}}(1+e)^{\frac{3}{2}} - 11F}{14ee} = 0{\cdot}0000{,}006,$$

&c. &c.

and thence

+	−
$A = 0{\cdot}6547{,}320$	$\frac{1}{2}B = 0{\cdot}0199{,}205$
$\frac{3}{2.4}C = 0{\cdot}0013{,}750$	$\frac{3.5}{2.4.6}D = 0{\cdot}0001{,}204$
$\frac{3.5.7}{2.4.6.8}E = 0{\cdot}0000{,}119$	$\frac{3.5.7.9}{2.4.6.8.10}F = 0{\cdot}0000{,}013$
$\frac{3.5.7.9.11}{2.4.6.8.10.12}G = 0{\cdot}0000{,}001$	
$+\,0{\cdot}6561{,}190$	$-\,0{\cdot}0200{,}422$
$-\,0{\cdot}0200{,}422$	

and $z = 0{\cdot}6360{,}768$; which value of z needs no correction.

We must now, in order to find the value of d, write the same values of e and y in Theorem IV, Art. 5, where we shall then have

$$A = \text{H. L.}\ (\sqrt{(1+e)} - \sqrt{e}),$$

$$B = \frac{-e\sqrt{\left(\frac{1}{e}+1\right)} - A}{2},$$

$$C = \frac{-ee\sqrt{\left(\frac{1}{e}+1\right)} - 3B}{4},$$

$$D = \frac{-e^3\sqrt{\left(\frac{1}{e}+1\right)} - 5C}{6},$$

$$E = \frac{-e^4\sqrt{\left(\frac{1}{e}+1\right)} - 7D}{8},$$

$$F = \frac{-e^5\sqrt{\left(\frac{1}{e}+1\right)} - 9E}{10},$$

&c. &c.

But, since the terms A, B, C, &c. are to be divided by e, e^3, e^5, &c. respectively, it will be best to divide them by these quantities, before we begin the arithmetical calculations; otherwise much unnecessary labour must be taken. The terms, then, being so divided, and the proper value of e being written for it, viz. $\sqrt{50}$, we shall have as below:

$$\frac{A}{e} = \frac{\text{H. L.}}{e}(\sqrt{(1+e)} - \sqrt{e} = -0.2410{,}905,$$

$$\frac{B}{e^3} = \frac{-\sqrt{\left(\frac{1}{e}+1\right)}}{2ee} - \frac{A}{2e^3} = -0.0082{,}728,$$

$$\frac{C}{e^5} = \frac{-\sqrt{\left(\frac{1}{e}+1\right)}}{4e^3} - \frac{3B}{4e^5} = -0.0006{,}314,$$

$$\frac{D}{e^7} = \frac{-\sqrt{\left(\frac{1}{e}+1\right)}}{6e^4} - \frac{5C}{6e^7} = -0.0000{,}607,$$

$$\frac{E}{e^9} = \frac{-\sqrt{\left(\frac{1}{e}+1\right)}}{8e^5} - \frac{7D}{8e^9} = -0.0000{,}065,$$

$$\frac{F}{e^{11}} = \frac{-\sqrt{\left(\frac{1}{e}+1\right)}}{10e^6} - \frac{9E}{10e^{11}} = -0.0000{,}007,$$

&c. &c.

And thence

$$\begin{array}{ll|ll}
 & + & & - \\
e\sqrt{\left(\frac{1}{e}+1\right)} & = 7\cdot5545{,}396, & +\frac{A}{2e} & = 0\cdot1205{,}452, \\
-\frac{B}{2.4e^3} & = 0\cdot0010{,}341, & +\frac{3C}{2.4.6e^5} & = 0\cdot0000{,}395 \\
-\frac{3.5D}{2.4.6.8e^7} & = 0\cdot0000{,}024, & +\frac{3.5.7E}{2.4.6.8.10e^9} & = 0\cdot0000{,}002 \\
\hline
 & +\,7\cdot5555{,}761 & & -\,0\cdot1205{,}849 \\
 & -\,0\cdot1205{,}849 & & \\
\end{array}$$

and $z = 7\cdot4349{,}912 - d$.

But, by the foregoing part of this article, $z = 0\cdot6360{,}768$; we therefore have $d = 7\cdot4349{,}912 - 0\cdot6360{,}768 = 6\cdot7989{,}144$.

22. With the value of a above given, viz. 7, we see a swift convergency, both in the ascending and in the descending series; but, if a were given $= \sqrt{3}$, (which is as small a value of a as need be used in these theorems, for this purpose, because if it were less than, or even $= \sqrt{3}$, the value of d might be computed by one series only, as was observed in Art. 15,) each of the series would converge but slowly, $\frac{1}{e}$, in this case, being $= \frac{1}{2}$; to remedy which, as the terms of each of the series have the signs + and − alternately, it would be expedient to compute a moderate number (from six to ten, as the case shall require,) of the initial terms of each, and then to transform the remainders into other series, which should converge by the powers of $\frac{1}{1+e}$, instead of the powers of $\frac{1}{e}$. This increase of convergency in the geometrical progression, assisted as it would be by the decrease of the coefficients of the new series, would enable us to get a result accurate enough for all common uses, by computing ten (or fewer) terms of each of the new series.

But, as the transformation now mentioned requires but a moderate skill in series, I shall, for the sake of brevity, omit examples of it, and proceed to

SECT. III. *Examples of the Use of the foregoing Theorems.*

23. My intention in this section is, to illustrate the use of the foregoing theorems by a few examples, selecting at the same time such of the theorems as are best adapted to the case in hand; by which, and attention to what was said in the first section, of the limits of the convergency of the several series, I hope the reader will be directed how to make a proper choice of theorems on all other occasions.

EXAMPLE I.

Let there be an hyperbola of which the semi-axes are 40 and 30 respectively, and the ordinate is 10; it is required to find the length of the arch from the vertex of the ordinate.

Since the conjugate semi-axis of this hyperbola is 30, we must, in order to fit the given numbers to our theorems, divide them all by 30; and then we shall have the corresponding dimensions of a similar hyperbola as follows; viz. the transverse semi-axis $= \frac{4}{3}$, the conjugate semi-axis $= 1$, and the ordinate $= \frac{1}{3}$. And the proper theorem to be used in this case is the second.

Writing, then, $\frac{4}{3}$ for a, and $\frac{1}{3}$ for y, in the IId Theorem, we have $ee = aa + 1 = \frac{25}{9}$, and

$$A = \frac{y}{2}\sqrt{(1+eeyy)} + \frac{1}{2e}\text{H. L.}\left(ey + \sqrt{(1+eeyy)}\right) = 0\text{'}3497,6260,$$

$$B = \frac{y(1+eeyy)^{\frac{3}{2}} - A}{4ee} = 0\text{'}0134,3234,$$

$$C = \frac{y^3(1+eeyy)^{\frac{3}{2}} - 3B}{6ee} = 0\text{'}0009,0892,$$

$$D = \frac{y^5(1+eeyy)^{\frac{3}{2}} - 5C}{8ee} = 0\text{'}0000,7272,$$

$$E = \frac{y^7(1+eeyy)^{\frac{3}{2}} - 7D}{10ee} = 0\text{'}0000,0632,$$

$$F = \frac{y^9(1+eeyy)^{\frac{3}{2}} - 9E}{12ee} = 0\text{'}0000,0058,$$

$$G = \frac{y^{11}(1+eeyy)^{\frac{3}{2}} - 11F}{14ee} = 0\text{'}0000,0005;$$

and thence

+	−
$A = 0\text{'}3497,6260,$	$\frac{1}{2}B = 0\text{'}0067,1617,$
$\frac{3}{2.4}C = 0\text{'}0003,4084,$	$\frac{3.5}{2.4.6}D = 0\text{'}0000,2273,$
$\frac{3.5.7}{2.4.6.8}E = 0\text{'}0000,0173,$	$\frac{3.5.7.9}{2.4.6.8.10}F = 0\text{'}0000,0014,$
$\frac{3.5.7.9.11}{2.4.6.8.10.12}G = 0\text{'}0000,0001,$	and the sum $= - 0\text{'}0067,3904.$
and the sum $= + 0\text{'}3501,0518,$	
$- 0\text{'}0067,3904,$	

difference of these sums is } $0\text{'}3433,6614 = z$, the length of the arch of an hyperbola, from the vertex to the ordinate, of which the transverse and conjugate semi-axes are $\frac{4}{3}$ and 1, and the ordinate $\frac{1}{3}$. And, since like parts of similar hyperbolas are to each other as their semi-axes, we shall have, by multiplying 0'3433,6614 by 30, the semi-conjugate of the hyperbola proposed, 10'3009,842 for the length required.

Having in this example shown how to adapt these theorems to hyperbolas that have a conjugate semi-axis different from 1,

it need not be repeated again. I shall therefore, in the remaining examples, show the convergency of these new series in most of the different cases that can occur.

EXAMPLE II.

24. Given $a = 1$, and $y = 1$, to find z.

This arch, it is obvious, may be computed by Theorem IVth, VIth, VIIth, and some others; but the VIIth is the proper one to be chosen on this occasion, as the series there given has the swifter convergency.

Writing, then, 1 for a, and 1 for y, in Theorem VII, we have (by Article 1,) $ee = aa + 1 = 2$, and, (by Art. 7,) $uu = 1 + eeyy = 3$; and then, (by Art. 9,)

$$+$$

$$u = \sqrt{3} = 1{\cdot}7320{,}5081,$$

$$-$$

$$\begin{aligned}
\tfrac{1}{2.3} u^{-3} &= 0{\cdot}0320{,}7501,\\
\tfrac{3}{2.4.7} u^{-7} &= 0{\cdot}0011{,}4554,\\
\tfrac{3.5}{2.4.6.11} u^{-11} &= 0{\cdot}0000{,}6750,\\
\tfrac{3.5.7}{2.4.6.8.15} u^{-15} &= 0{\cdot}0000{,}0481,\\
\tfrac{3.5.7.9}{2.4.6.8.10.19} u^{-19} &= 0{\cdot}0000{,}0038,\\
\tfrac{3.5.7.9.11}{2.4.6.8.10.12.23} u^{-23} &= 0{\cdot}0000{,}0003;
\end{aligned}$$

sum of the neg. terms $- 0{\cdot}0332{,}9327$;

sum of the series $+ 1{\cdot}6987{,}5754$;

correction of the fluent $- 0{\cdot}5990{,}7012 = -d$, (by Art. 17;)

the difference of which is $+ 1{\cdot}0996{,}8742 = z$.

EXAMPLE III.

25. Given $a = 1$, and $y = \sqrt{(1000000 - 1)} = 999{\cdot}9995$ nearly, to find z.

This arch, it is very obvious, may be computed by Theorem IIId, IVth, VIth, and VIIth, the series in each of them converging, in this case, very swiftly. And it may be computed also by the IXth; but the proper Theorem to be used in this case is the VIIth.

Now, since ee is $= 2$, and $y = \sqrt{(1000000 - 1)}$, we have (by Article 7,) $u = \sqrt{(2yy + 1)} = \sqrt{(2000000 - 1)} = \sqrt{2} \times \sqrt{(1000000 - \frac{1}{2})} = \sqrt{2} \times \left(1000 - \frac{1}{4.1000}\right)$ very nearly, $= 1000\sqrt{2} - \frac{\sqrt{2}}{4000} = 1414{\cdot}2132088$, which may be taken for the value of the whole series, since $\frac{1}{6}u^{-3}$, the second term of it, does not give a 1 in the tenth place of decimals. If, therefore, from $u = 1414{\cdot}2132088$, we subtract $d = 0{\cdot}5990701$, (by Article 17,) we shall have $z = 1413{\cdot}6141387$,* the length required.

EXAMPLE IV.

26. Let a be given $= 7$, and $y = 10$, to find z.

This Example may be computed by Theorem IIId, IVth, VIth, and some others; the VIth is to be chosen rather than the IIId, and the IVth rather than the VIth.

* The computation of the value of z, in this example, is the problem alluded to in the Introduction to this Paper, which first turned my thoughts to the subject of it, in the year 1770. In the next year, two answers were given to it, by two persons of good reputation for their skill in mathematics, one of them making $z = 1414{\cdot}2132088$, the other, $z = 1413{\cdot}8921$. These two are the only solutions of this problem that I know of; and, if my calculation be right, both are erroneous.

Now, if 10 be written for y in Theorem IVth, we shall have

$$A = \text{H. L.}\ \frac{\sqrt{(yy+1)}-1}{y} = -0\cdot0998{,}341,$$
$$B = \frac{-\sqrt{(yy+1)}}{2yy} - \frac{A}{2} = -0\cdot0003{,}323,$$
$$C = \frac{-\sqrt{(yy+1)}}{4y^4} - \frac{3B}{4} = -0\cdot0000{,}020;$$

of which terms, two only are wanted to obtain a result true to seven places of figures. And then, ee being $= aa + 1 = 50$, we have

+		−	
$e\sqrt{(yy+1)} =$	$71\cdot0633{,}520,$	$+\frac{1}{2e}A =$	$0\cdot0070{,}593,$
$-\frac{1}{2.4e^3}B =$	$0\cdot0000{,}001,$	$-d =$	$6\cdot7989{,}144,$ (Art. 21,)
sum of the posit. terms	$71\cdot0633{,}521;$	the sum	$-6\cdot8059{,}737;$
neg. term, and correct.	$-6\cdot8059{,}737;$		

the difference is $+64\cdot2573{,}748 = z$.

Example V.

27. Let a be given $= \frac{1}{2}$, and $y = 10$, to find z.

This example may be computed by Theorem IIId, IVth, Vth, VIth, VIIIth, and IXth; of which the IVth, Vth, and IXth, are more eligible than the other three. I make choice of the fourth, on account of the facility of the computation by it, with the present value of y.

Now, by writing 10 for y in Theorem IV, we shall have (as in the preceding example,)

$$A = \text{H. L.}\ \frac{\sqrt{(yy+1)}-1}{y} = -0\cdot0998{,}341,$$
$$B = \frac{-\sqrt{(yy+1)}}{2yy} - \frac{A}{2} = -0\cdot0003{,}323,$$
$$C = \frac{-\sqrt{(yy+1)}}{4y^4} - \frac{3B}{4} = -0\cdot0000{,}020;$$

and then, ee being $= aa + 1 = \frac{5}{4}$, we have

+	−
$e\sqrt{(yy+1)} = 11\cdot2361{,}025,$	$\frac{A}{2e} = 0\cdot0446{,}472,$
$-\frac{1}{2\cdot4e^3}B = 0\cdot0000{,}297,$	$\frac{3}{2\cdot46e^5}C = 0\cdot0000{,}001,$
sum of affirm. terms $+\ 11\cdot2361{,}322;$	$-d = 0\cdot1803{,}793,$ (Art. 15,)
neg. terms and corr. $-\ 0\cdot2250{,}266;$	the sum is $-\ 0\cdot2250{,}266.$
the difference is $= 11\cdot0111{,}056,$	
which is $= z.$	

28. Having now produced series, of good convergency, for computing the length of the arch from the vertex to the ordinate, (and consequently any portion of such an arch,) of any conical hyperbola, I shall conclude this Paper with a few remarks: reserving some other theorems which I have discovered for the purpose, till I shall have found an opportunity to describe nearly an equal number of theorems, which I have long had by me, for the Rectification of the Ellipsis.

The utility of hyperbolic and elliptic arches, in the solution of various problems, and particularly in the business of computing fluents, has been shown by those eminent mathematicians, McLAURIN, SIMPSON, and LANDEN; the last of whom hath written a very ingenious paper on hyperbolic and elliptic arches, which was published in the Ist volume of his *Mathematical Memoirs*, in the year 1780. I have indeed heard, that some improvement in the rectification of the ellipsis and hyperbola had been produced, and some of the same theorems discovered, by a learned Italian, many years before Mr. LANDEN'S *Mathematical Memoirs* were published; but, as Mr. LANDEN has declared that he had never seen nor heard any thing of that work, and as various instances are to be found of different men

discovering the same truth, without any knowledge of each other's works, I see no reason for disbelieving him. But I have seen no writings on this subject which contain any thing more than what is very common, besides those of the three gentlemen above mentioned, and Dr. WARING's *Meditationes Analyticæ;* and, while I have no inclination to detract from their merits, I may be allowed to say that I have borrowed nothing from their works.

29. With respect to Dr. WARING, (who was well known to be a profound mathematician, and I can testify that he was a good-natured man,) he has given, in page 470 of his *Meditationes Analyticæ*, (published in 1776,) these two series, as expressions of the length of an arch of an equilateral hyperbola; viz.

"Arcus hyperbolicus exprimi possit per seriem $-\frac{1}{x}+\frac{1}{2\times3}x^3$
"$-\frac{1}{2^2.2\times7}x^7+\frac{1.3}{2^3.2.3\times11}x^{11}-\frac{1.3.5}{2^4.2.3.4\times15}x^{15}+\frac{1.3.5.7}{2^5.2.3.4.5\times19}x^{19}$,
"&c. ubi x denotat abscissam ad asymptoton."

"Si vero requiratur descendens series, tum erit $x-\frac{1}{2\times3}x^{-3}$
"$+\frac{1}{2^2.2\times7}x^{-7}-\frac{3^*}{2^3.2.3\times11}x^{-11}$, &c. quæ, quoad coefficientes
"attinet, prorsus eandem observat legem ac præcedens."

30. These series, as they now stand, are of little use. But, if proper corrections were applied to them, (which may easily be done from what has been shewn in this Paper, and in my *Mathematical Essays,)* and the first of them were transformed into another series converging by the powers of $\frac{x^4}{1+x^4}$, they would become very useful for computing any arch of an equilateral

* In the original, this term is erroneously printed, there being a 1 in the numerator, instead of a 3.

hyperbola, when the abscissa is taken on the asymptote. This I thought it might be proper to remark, that the less experienced readers of this Paper might not be misled by so great an authority as that of Dr. WARING. Whether or not he ever corrected these oversights in any of his subsequent publications, I cannot ascertain, for want of books.